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Real Analysis [Depends on sequence theorems]

Q.1. prove that a convergent sequence determines its limit uniquely.

Soln: proof:→ Let $\{u_n\}$ be a convergent sequence.

If possible, let l and l' be its two limits.

Since $\{u_n\}$ is convergent, we have a positive integer m depending on ϵ, an arbitrary positive number however small, such that

$$|u_n - l| < \frac{\epsilon}{2}, \text{ for all } n \geq m.$$

and $|u_n - l'| < \frac{\epsilon}{2}$, for all $n \geq m$.

Now,
$$\begin{aligned} |l - l'| &= |(u_n - l') - (u_n - l)| \\ &\leq |u_n - l'| + |u_n - l| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for all } n \geq m. \end{aligned}$$

But ϵ is arbitrary positive number however small.

Hence, ultimately $l = l'$

Thus, the limit is unique. proved.

Q.2. Prove that every convergent sequence is bounded.

proof: Let the sequence $\{a_n\}$ be convergent. So, it must tend to a limit, say l.

Then, by the definition of limit of a sequence, given $\epsilon > 0$, however small, there exists a positive integer m, such that

$$|a_n - l| < \epsilon, \text{ for all } n \geq m$$

or, $l - \epsilon < a_n < l + \epsilon$, for all $n \geq m$.

Let k and K be the least and the greatest of

$$a_1, a_2, a_3, \dots a_{m-1}, l-\epsilon, l+\epsilon$$

Then, $k \leq a_n \leq K$, for all $n \in N$.

Hence, the sequence $\{a_n\}$ is bounded.

proved

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Q.3. Prove that a monotone increasing sequence tends to its upper bound.

Proof: Let $\{a_n\}$ be a monotone increasing sequence whose (least) upper bound is M. Then, two cases arise.

(I) M is finite (II) M is infinite.

When M is finite: By definition of least upper bound.

(I) $a_n \le M$, for all n. (II) $a_n + \epsilon > M$, for at least one n.

Let this be true for $n = m$.

Then $a_m + \epsilon > M$, i.e. $a_m > M - \epsilon$ ——— (2)

But the sequence is monotone increasing. So,

$$a_n \ge a_m, \text{ if } n \ge m \quad \text{——— (3)}$$

∴ From (2) and (3), $a_n > M - \epsilon$, for all $n \ge m$

Again, from (I), $a_n \le M$, for all n. So, $a_n < M + \epsilon$, for all n and hence, also for $n \ge m$.

∴ Combining these, $M - \epsilon < a_n < M + \epsilon$, for $n \ge m$.

∴ $|a_n - M| < \epsilon$, for $n \ge m$.

∴ $\{a_n\}$ tends to M.

When M is infinite: Then from (2), $a_m > M - \epsilon = N$, for $n \ge m$.

when N is arbitrary large.

But $a_n \ge a_m > N$, for $n \ge m$.

∴ $\{a_n\}$ tends to ∞ and so to the upper bound ∞.

Q.4: Define sequence with examples;

Soln: Sequence:→ If, to each positive integer 1, 2, 3, ... there corresponds a definite real number u_n, then the numbers $u_1, u_2, u_3, u_4, \ldots, u_n, \ldots$ are said to form a sequence.

A sequence of ~~limits~~ elements $u_1, u_2, u_3, \ldots u_n, \ldots$ is denoted by $\{a_n\}$ or by $\{u_n\}$.

A sequence is said to be finite or infinite according as the number of elements is finite or infine.

We shall consider only the infinite sequences.

For examples: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots$ and $2, 4, 6, \ldots, 2n, \ldots$